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LETTER TO THE EDITOR

Asymptotic form of the spectral dimension of the Sierpinski gasket type of fractals

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Abstract. We have studied the spectral dimension $\tilde{d}$ of an infinite class of fractals. The first member ($b=2$) of the class is the two-dimensional Sierpinski gasket, while the last member ($b=\infty$) appears to be a wedge of the ordinary triangular lattice. By studying the electric resistance of the fractals we have been able to calculate exact values of $\tilde{d}$ for the first 200 members of the class. An analysis of the obtained data reveals that for large b the spectral dimension should approach the upper limit of 2 according to the formula $\tilde{d} \simeq 2 - \text{constant} (\ln b)^{\beta}$, where β is not larger than one. This result implies, among other things, that the scaling exponents of the resistivity and diffusion constant should logarithmically vanish at the fractal-lattice crossover.

Recently, there has been a good deal of interest in the quantities which determine the dynamical properties of fractal structures, such as percolation clusters. Alexander and Orbach (1982) were the first to point out that at least three dimensions are required to describe the linear problems (classical diffusion, spectrum of low-energy excitations and electrical conductivity) on fractals (see, for instance, Alexander 1983). The first two dimensions, i.e. the embedding (Euclidean) dimension d and the fractal dimension $\bar{d}$, determine the spatial and geometrical features of fractals, whereas the third one, the so-called spectral dimension $\tilde{d}$, controls the relevant physical properties. Thus, for instance, the density of states is given by the power law $\rho(\omega) \sim \omega^{\tilde{d}-1}$ for low frequencies ω (Dhar 1977, Alexander and Orbach 1982, Rammal and Toulouse 1983, Gefen *et al* 1983). Since there are many other dynamical quantities that can be directly related to $\tilde{d}$, it should be useful to attain a complete knowledge of $\tilde{d}$ for certain classes of fractals.

In this letter we study the spectral dimension of an infinite class of fractals, whose first member (labelled by $b=2$) is the two-dimensional Sierpinski gasket, while its last member ($b=\infty$) appears to be a wedge of the ordinary triangular lattice. The spectral dimensions for the first fifteen members of the class have been previously calculated by Hilfer and Blumen (1984) using the master equation approach. However, the known set of values of $\tilde{d}$ has not been sufficient to deduce an asymptotic behaviour of $\tilde{d}$ when $b \to \infty$, that is to say the behaviour of $\tilde{d}$ at the fractal-lattice crossover. Here we examine the electric DC conductivity of the fractals and demonstrate that by this approach one can find $\tilde{d}$ for a much larger set of members of the class studied. In particular, we have calculated $\tilde{d}$ for all values of b up to 200, and on the basis of the obtained data we have been able to infer an asymptotic form of the function $\tilde{d}(b)$.

The spectral dimension $\tilde{d}$ of a fractal can be related to the exponent of the power law $R_L \sim L^{\zeta}$ which describes the fractal electric resistance R_L between two points that lie the scale length L apart (see, for instance, Rammal *et al* 1984, Given and Mandelbrot 1983). The relation is of the form

$$\tilde{d} = 2\bar{d}/(\bar{d}+\zeta). \tag{1}$$

In order to calculate ζ for the class of fractals under study we recall that each member of the class can be obtained from a particular generator $G(b, 2)$, which appears to be an equilateral triangle that contains b^2 identical smaller triangles of unit side length (Hilfer and Blumen 1984). Since at the nth stage of growing of a fractal (see figure 1 of Elezović *et al* (1987)) the generator is enlarged by b^n and filled with the stage $(n-1)$ structure, it follows that the fractal resistance R_{nb} and $R_{(n-1))b}$ are related by

$$R_{nb} = b^{\zeta} R_{(n-1)b} \tag{2}$$

which reduces to

$$R_b = b^{\zeta} R_1 \tag{3}$$

where R_b is the resistance of the generator and R_1 is the resistance of a unit triangle. If each bond of each unit triangle carries a resistor with a unit resistance, then $R_1 = \frac{2}{3}$ and relation (3) implies

$$\zeta = \ln(\tfrac{3}{2}R_b)/\ln b. \tag{4}$$

Accordingly, finding the exponent ζ, and consequently finding the spectral dimension $\tilde{d}$, is converted, due to the fractal self-similarity, to an evaluation of the generator resistance R_b.

One can apply various techniques to evaluate the electrical resistance R_b. We have found that the successive application of the star-triangle transformation is the most effective (see figure 1). This approach enables us to reduce every generator $G(b, 2)$, conceived as a complex electrical circuit, to a simpler circuit represented by $G(b-1, 2)$, and by further reductions we can finally reach a circuit in the shape of a unit triangle. The corresponding iterative transformations of the resistances can be easily found, and by applying them one can find R_b in the form of rational numbers, at least for small b. Thus for $2 \leqslant b \leqslant 5$ we regained numbers found by Given and Mandelbrot (1983) (see their table 1 and note that their R is just our ratio R_b/R_1), and for $b=6$ we found that $R_6 = 19\,015\,038/9294\,075$. However, for larger b the application of the resistance transformations, although straightforward, becomes strenuous, and, for this reason, we have computerised the whole procedure. In table 1 we display a representative set of the values of $\tilde{d}$ obtained, via formulae (1) and (4), from the computed values of R_b. One can infer from table 1 that the values of $\tilde{d}$ less than two increase very slowly when b increases.

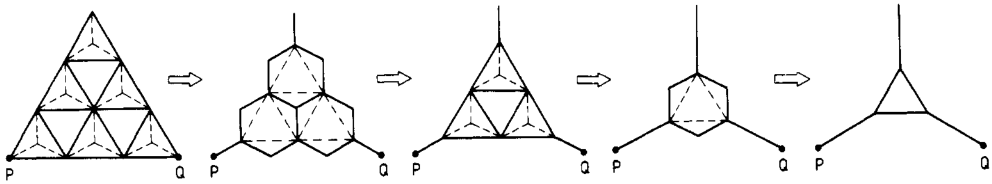

Figure 1. Reduction of the $b=3$ fractal generator according to the star-triangle transformation. It is assumed that at the beginning each bond of each unit triangle of the generator carries a resistor with a unit resistance. An electric DC current is sent into the vertex P and taken out of the vertex Q.

Table 1. A sample set of values of the spectral dimension for the Sierpinski gasket type of fractals.

b	$\tilde{d}$	b	$\tilde{d}$
10	1.4990	110	1.6217
20	1.5424	120	1.6250
30	1.5644	130	1.6279
40	1.5787	140	1.6306
50	1.5891	150	1.6330
60	1.5972	160	1.6352
70	1.6038	170	1.6373
80	1.6093	180	1.6392
90	1.6140	190	1.6411
100	1.6181	200	1.6428

It has been observed (Rammal and Toulouse 1983) and argued (Hattori *et al* 1986) that, for finitely ramified fractals (including those which are embedded in the Euclidean spaces with $d > 2$), $\tilde{d}$ cannot be larger than two. The spectral dimension of the Sierpinski type of fractals, for fixed b, approaches the upper limit of two when $d \to \infty$. As a matter of fact, one can verify that the known closed-form expressions for the spectral dimension, found for $b = 2$ (Rammal and Toulouse 1983) and $b = 3$ (Hilfer and Blumen 1984), imply the following asymptotic law:

$$\tilde{d} \simeq 2 - c/d \ln d \qquad d \to \infty \tag{5}$$

where c is equal to 4 and 2 for $b = 2$ and $b = 3$, respectively. In the opposite case, that is to say in the case of fractals with fixed d and arbitrary b, we may expect that their spectral dimensions approach the upper limit of two when their fractal dimensions $\bar{d}$ approach the embedding dimension d. From the general formula (Hilfer and Blumen 1984)

$$\bar{d} = \ln \binom{b+d-1}{d} (\ln b)^{-1} \tag{6}$$

it follows that $\bar{d} \to d$ when $b \to \infty$. Therefore, in the case under study, we can accept that table 1 represents a subsequence of an infinite sequence of numbers which tend to two and try to find an asymptotic form of the general term of the sequence when $b \to \infty$.

Proceeding from (5) we could assume an analogous asymptotic form in the case of fixed d:

$$\tilde{d} \simeq 2 - c/b \ln b \qquad b \to \infty. \tag{7}$$

However, such a form would lead to a paradoxical result appearing, in that for some very large b the corresponding fractals may have $\tilde{d} > \bar{d}$ as $\tilde{d}$ given by (7) would be closer to two than the fractal dimension which is, in accordance with (6), given by $\bar{d} \simeq 2 - \ln 2/\ln b \to \infty$. The inequality $\tilde{d} > \bar{d}$ would be paradoxical since it would, in conjunction with the formula $\theta = 2(\bar{d}/\tilde{d} - 1)$ (see, for instance, Alexander 1983), imply that the scaling exponent θ of the coefficient of diffusion could be negative, which would in turn bring about a superanomalous diffusion. For the same reason, any power-law correction for $\tilde{d}$ should not be acceptable. The latter conclusion can be corroborated by a numerical analysis.

We have performed a least-squares fitting of the whole set ($2 \leqslant b \leqslant 200$) of our exact data to the power-law asymptotic formula

$$\tilde{d} \simeq 2 - A/b^{\alpha} \tag{8}$$

and to the logarithmic asymptotic formula

$$\tilde{d} \simeq 2 - B/(\ln b)^{\beta} \tag{9}$$

where A, α, B and β are the fitting constants. Surprisingly, we have found that for a suitable choice of constants both formulae reproduce the exact data seemingly well. However, grouping our data for larger b into successive intervals of 21 and performing independent fitting for each interval, we have found that the mean-square deviations D of the power-law asymptotic formula are persistently larger than the corresponding deviations of the logarithmic asymptotic formula (see figure 2). In addition, the behaviour of the fitting constants is indicative of the incompatibility of the power law (8). In fact, table 2 reveals that values of α are small and monotonically decreasing, which reflects the fact that the power-law correction should eventually be extremely weak. This necessity and the fact that values of the exponent β increase, together with the finding that a similar situation occurs if one performs numerical analysis of the first 200 data for the fractal dimension, lead us to conclude that the asymptotic behaviour of the spectral dimension should be of the form

$$\tilde{d} \simeq 2 - B/(\ln b)^{\beta} \qquad b \to \infty \tag{10}$$

where $\beta \leqslant 1$. If it turns out that $\beta = 1$, then the constant B cannot be smaller than $\ln 2$, for $B < \ln 2$ would bring about the incorrect inequality $\bar{d} < \tilde{d}$ for some large b.

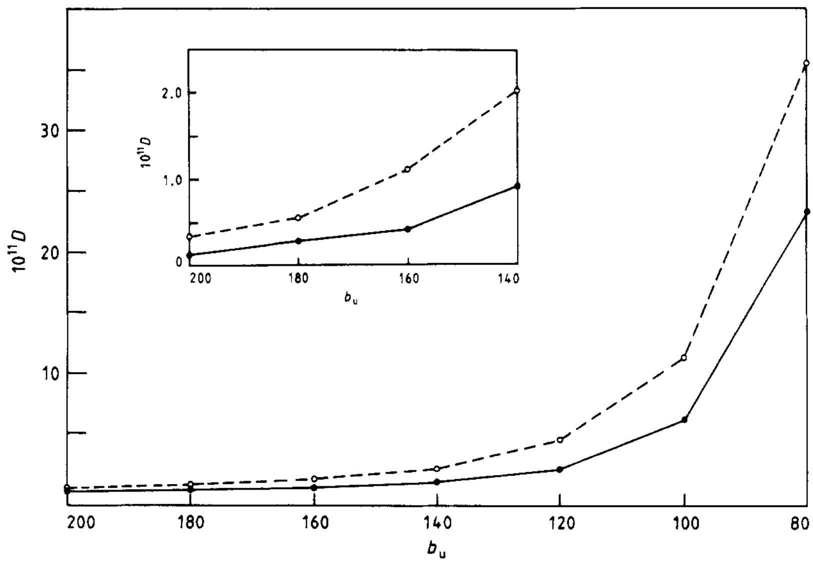

Figure 2. Plot of the mean-square deviations D of the spectral dimension $\tilde{d}$ evaluated according to equations (8) and (9), with the constants given in table 2, from the exact values of $\tilde{d}$. The upper bounds of the intervals given in table 2 are designated as b_u. The full circles correspond to the logarithmic asymptotic formula, whereas the open circles correspond to the power-law asymptotic formula (the full and broken curves serve as a guide to the eye). The insert represents the last four data in a frame with the enlarged vertical scale.

Table 2. Values of the constants which furnish the best fits of the calculated spectral dimensions (1) to equations (8) and (9).

b	A	α	B	β
$40 \leq b \leq 60$	0.633 56	0.110 65	0.739 26	0.430 62
$60 \leq b \leq 80$	0.620 96	0.105 73	0.757 62	0.448 08
$80 \leq b \leq 100$	0.611 71	0.102 29	0.770 70	0.459 69
$100 \leq b \leq 120$	0.604 45	0.099 70	0.780 79	0.468 22
$120 \leq b \leq 140$	0.598 49	0.097 63	0.789 05	0.474 94
$140 \leq b \leq 160$	0.593 43	0.095 91	0.795 91	0.480 36
$160 \leq b \leq 180$	0.589 05	0.094 45	0.801 77	0.484 88
$180 \leq b \leq 200$	0.585 25	0.093 20	0.807 11	0.488 91

Accepting this conclusion one may ponder on the second-order term in (10). We mention here that we have tried to add to the right-hand side of equation (10) either constant/$b \ln b$ or constant/b and found that both possibilities can reproduce exact data plausibly well (the same happens if we try constant/$b^{1/2}$ instead). Thus we cannot see any reason to suggest anything more than (10). Indeed, the result (10) alone seems to be interesting and stimulating. It implies that the scaling exponent ζ of the resistivity and the scaling exponent θ of the diffusion constant vanish logarithmically at the fractal-lattice crossover. Finally, it should stimulate further research with an aim to disclose the behaviour of similar quantities in the case of the finitely ramified fractals embedded in three-dimensional Euclidean space.

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